Analysis 1

12 December 2023

Warm-up: Give the *seventh* derivative of $f(x) = \sin(2x)$.

This can be written $\dfrac{\mathrm{d}^7 f}{\mathrm{d}x^7}$ or f''''''' or $f^{(7)}$.

Answer: $-128\cos(2x)$

You should memorize these!

Last time

Function	Derivative
x^p	$p\,x^{p\text{-}1}$
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\ln(x)$	$\frac{1}{x}$

Function	Derivative
$c \cdot f(x)$	$c \cdot f'$
$f(x) \cdot g(x)$	$f\,g' + f'g$
$f(x) + g(x)$	$f' + g'$
$f(g(x))$	$f'(g(x)) \cdot g'(x)$

In the top rows, p and c must be constants.

Last time

Function	Derivative
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
a^x	$a^x \ln(a)$
$\tan(x)$	$\sec(x)^2$

Here a must be constant.

Function	Derivative
$\frac{f(x)}{g(x)}$	$\frac{fg' - fg'}{g^2}$

These do not need to be memorized because they can be figured out from the previous rules. But you can solve some tasks faster if you do memorize these.

Applications of derivatives

In this class:

- finding tangent lines
- approximating function values ← we'll revisit this today
- determining when a graph is increasing or decreasing
- finding critical points
- finding absolute minima and maxima
- finding local minima and maxima (1st Derivative Test or 2nd D. T.)
- determining when a graph is concave up or concave down
- calculating certain limits ← new today

There are *hundreds* of other applications of derivatives in science/engr.

https://chat.openai.com/chat

TH

Find the local minimum and maximum of x^3 - 3x^2 - 9x.

To find the local minimum and maximum of a function, you can try to find the points where the derivative of the function is equal to 0, because these points are points of inflection. If the derivative is positive on one side of these points and negative on the other side, then the function is increasing on one side and decreasing on the other side, which means that these points are local maxima. If the derivative is negative on one side and positive on the other side, then the function is decreasing on one side and increasing on the other side, which means that these points are local minima.

The derivative of x^3 - 3x^2 - 9x is 3x^2 - 6x - 9. Setting this equal to 0, we find that the points where the derivative is 0 are x = 0 and x = 3/2.

We can then use the second derivative test to determine whether these points are local minima or local maxima. The second derivative of x^3 - 3x^2 - 9x is 6x - 6. Evaluating the second derivative at x = 0 and x = 3/2, we find that the second derivative is -6 at x = 0 and 6 at x = 3/2.

Since the second derivative is negative at x = 0 and positive at x = 3/2, this tells us that x = 0 is a local maximum and x = 3/2 is a local minimum.

So, the local minimum of the function x^3 - 3x^2 - 9x is at x = 3/2, and the local maximum is at x = 0.

Points where the derivative equals 0 are *critical* points. *Inflection* points are points where the second derivative changes sign.

You have to say which is left side and which is right side to know if the point is a local minimum or a local maximum.

Setting 3x^2 - 6x - 9 equal to zero should give **x = -1** and **x = 3**, not 0 and 3/2.

At x = 3/2, the value of 6x - 6 is 3, not 6.

I asked ChatGPT some other analysis tasks, and it usually described the *correct method* to use (e.g., set the derivative equal to zero and check whether f'' is positive or negative) but contained *several errors*.

Limits (again)

Previously, we did these kinds of limits by hand:

- For a continuous function, just plug in the value.

$$\lim_{t \to 5} \frac{t^2 + t - 3}{t - 3} = \frac{25 + 10 + 3}{5 - 3} = 19.$$

- Use algebra to re-write the formula.

$$\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4} = \lim_{x \to 4} \frac{(x-4)(x+3)}{x - 4} = \lim_{x \to 4} (x + 3) = 7.$$

- Recognize a derivative setup.

$$\lim_{h \to 0} \frac{(2+h)^{10} - 1024}{h} = f'(2) \text{ for } x^{10}\text{, which is } 10(2^9) = 5120.$$

It's common to see limits that look like "$\frac{0}{0}$" if you just plug in the value (derivatives always look this way). There is a nice trick we can use for these:

L'Hôpital's Rule

also spelled L'Hospital

If f and g are differentiable near $x = a$, and $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists, and either

- $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, or
- $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, ← technically four cases (++, +−, −+, −−)

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Also true for $\lim_{x \to a^+}$ and $\lim_{x \to a^-}$. Note that $\frac{f}{g} \neq \frac{f'}{g'}$; it is only the limits that are equal.

Example 1:

Find $\lim\limits_{x \to 1} \dfrac{x^3 - x^2 + x - 1}{x^4 - x^3 + 4x - 4}$.

L'Hôpital's Rule

For $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$, use $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$.

First, check that $\dfrac{f}{g}$ is $\dfrac{0}{0}$. Then use L'Hôpital.

$$\frac{(1)^3 - (1)^2 + (1) - 1}{(1)^4 - (1)^3 + 4(1) - 4} = \frac{0}{0}$$

$$\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x^4 - x^3 + 4x - 4} = \lim_{x \to 1} \frac{3x^2 - 2x + 1}{4x^3 - 3x^2 + 4} = \frac{3-2+1}{4-3+4} = \boxed{\frac{2}{5}}$$

L'H: For $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$, use $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

Example 2.

Calculate $\lim\limits_{x\to 0}\dfrac{\sin(x)-x}{x^3}$.

Use L'Hôpital 3 times.

Final answer: $\dfrac{-1}{6}$

L'H: For $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, use $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$.

Example 3.
Calculate $\lim\limits_{x\to 0^+} x\ln(x)$.

This is "$0 \cdot -\infty$".

Re-write as $\frac{\ln(x)}{1/x}$ to get $\frac{-\infty}{\infty}$. Then use L'H.

Final answer: 0

L'H: For $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$, use $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Example 4.

Find $\displaystyle\lim_{x\to 4}\frac{4x^2-28}{x^2-4}$.

L'Hôpital's Rule does not apply here!

Just plug x=4 into the fraction.

Answer: 3

Polynomials

For a function $f(x)$ and a number a, these two problems are the same:

- Find the function $P(x)$ for which $y = P(x)$ is the tangent line to $y = f(x)$ at $x = a$.

- Find the linear function $P(x)$ for which $P(a) = f(a)$ and $P'(a) = f'(a)$.

Example: Find $P(x)$ such that $P(0) = f(0)$ and $P'(0) = f'(0)$ for

$$f(x) = \ln(3x+1).$$

$f(0) = \ln(1) = 0$

$f'(x) = 1/(3x+1) \cdot 3$

$\quad = 3/(3x+1)$

$f'(0) = 3$

$$\boxed{P(x) = 3x}$$

What if we want $P''(0) = f''(0)$ also?

$$f''(x) = \frac{(3x+1)(0) - 3(3)}{(3x+1)^2} = -9/(3x+1)^2$$

$f''(0) = -9$

But how do we use this?

$y = \ln(3x+1)$ in blue

$y = 3x + \mathbf{k}\, x^2$ in orange

$f''(0) = -9$ from before

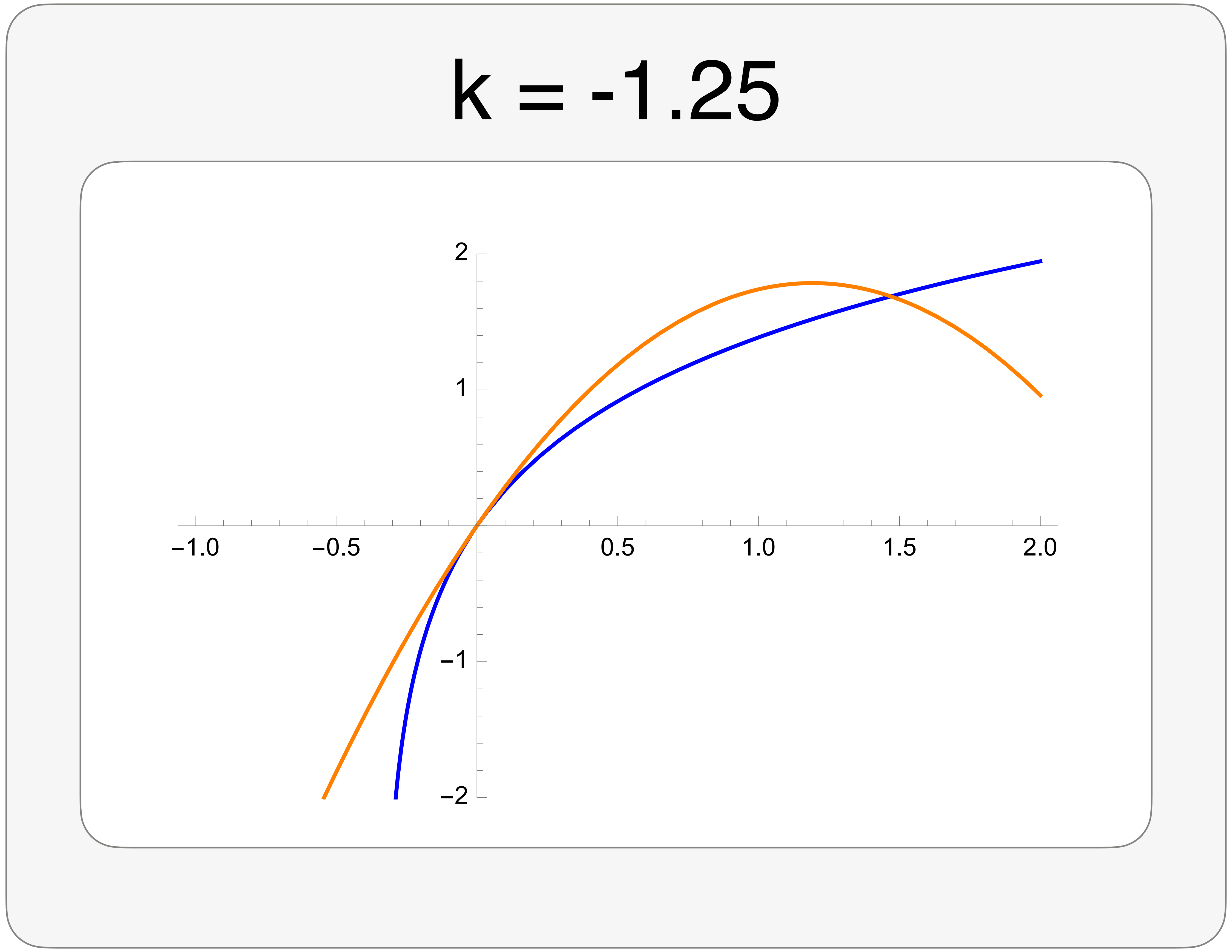
$P(x) = 3x + kx^2$

$P'(x) = 3 + 2kx$

$P''(x) = 2k$

$P''(0) = 2k$

We want $P''(0) = f''(0)$.
To get $2k = -9$, we need $k = -9/2$.



Taylor polynomials

What we have just done is an example of a "Taylor polynomial".

For $f(x) = \ln(3x+1)$, the Taylor polynomial of degree 2 around $x = 0$ is

$$3x - \frac{9}{2}x^2.$$

The degree 3 Taylor polynomial is $3x - \dfrac{9}{2}x^2 + 9x^3$.

The degree 4 Taylor polynomial is $3x - \dfrac{9}{2}x^2 + 9x^3 + \dfrac{81}{4}x^4$.

How can we find these in general?

Taylor polynomials

We want a way to find a polynomial

$$P(x) \quad = \quad A + Bx + Cx^2 + Dx^3 + \cdots + Zx^n$$

that has the same derivatives as $f(x)$, up to the n^{th} derivative.

It can also be helpful to think of an "infinite degree polynomial" and say

$$f(x) \quad = \quad A + Bx + Cx^2 + Dx^3 + \cdots$$

exactly. For example, you may have seen

$$1 + r + r^2 + r^3 + r^4 + \cdots \quad = \quad \frac{1}{1-r}$$

called a "geometric series" in another class. This is a "Taylor *series*".

Taylor series for cos(x)

If we want $\cos(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$ for some coefficients $c_0, c_1, \ldots,$ how could we find the values of the coefficients?

$$\cos(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$$

- If we plug in $x = 0$, this becomes $\cos(0) = c_0 + 0 + 0 + \cdots,$ so c_0 must be $\cos(0)$, which is 1.

Take a derivative of the entire equation above. Then plug in $x = 0$ again:

$$-\sin(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots \qquad -\sin(0) = c_1 \qquad c_1 = 0$$

Taylor series for cos(x)

If we want $\cos(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$ for some coefficients $c_0, c_1, \ldots,$ how could we find the values of the coefficients?

$$\cos(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \qquad c_0 = \cos(0)/0!$$

$$-\sin(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots \qquad c_1 = -\sin(0)/1!$$

$$-\cos(x) = 2c_2 + (3\cdot 2)c_3 x + (4\cdot 3)c_4 x^2 + \cdots \qquad c_2 = -\cos(0)/2!$$

$$\sin(x) = (3\cdot 2\cdot 1)c_3 + (4\cdot 3\cdot 2)c_4 x + \cdots \qquad c_3 = \sin(0)/3!$$

$$\cos(x) = (4\cdot 3\cdot 2\cdot 1)c_4 + (5\cdot 4\cdot 3\cdot 2)c_5 x + \cdots \qquad c_4 = \cos(0)/4!$$

$$-\sin(x) = \underbrace{(5\cdot 4\cdot 3\cdot 2\cdot 1)}_{\text{This is 5! (the “factorial” of 5).}} c_5 + \cdots \qquad c_5 = -\sin(0)/5!$$

Taylor series for f(x)

If we want $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$ then

$$c_0 = f(0)/0!$$

$$c_1 = f'(0)/1!$$

$$c_2 = f^{(2)}(0)/2!$$

$$c_3 = f^{(3)}(0)/3!$$

$$c_4 = f^{(4)}(0)/4!$$

$$c_5 = f^{(5)}(0)/5!$$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

The **Taylor series** **around $x = 0$** for $f(x)$ is

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This is $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$ forever.

For “analytic” functions (which includes many common functions), the Taylor series is exactly $f(x)$.

The **degree** N **Taylor polynomial** **around $x = 0$** for $f(x)$ is

$$P(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n.$$

The only difference is ∞ instead N at the top of the Σ.

The **Taylor series around $x = a$** for $f(x)$ is

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is $f(a) + f'(x-a)x + \frac{f''(x-a)}{2!}x^2 + \cdots$ forever.

For “analytic” functions (which includes many common functions), the Taylor series is exactly $f(x)$.

The **degree N Taylor polynomial around $x = a$** for $f(x)$ is

$$P(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example: Find the degree 4 Taylor polyn. for $f(x) = \sin(x)$ around $x = 0$.

$$P(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k$$

$f(x) = \sin(x)$ $\quad f(0) = \sin(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = \cos(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = -\sin(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -\cos(0) = -1$

$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(0) = \sin(0) = 0$

$$= \frac{\sin(0)}{0!} + \frac{\cos(0)}{1!} x + \frac{-\sin(0)}{2!} x^2 + \frac{-\cos(0)}{3!} x^3 + \frac{\sin(0)}{4!} x^4$$

$$= \frac{0}{1} + \frac{1}{1} x + \frac{0}{2} x^2 + \frac{-1}{6} x^3 + \frac{0}{24} x^4$$

$$= \boxed{x - \frac{1}{6} x^3}$$

Common Taylor series

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

Task: Find the degree 8 Taylor polynomial for $f(x) = \cos(x^2)$.

- Option 1: Compute lots of derivatives.

$$f'(x) = -2x\sin(x^2) \qquad f'(0) = 0$$
$$f''(x) = -4x^2\cos(x^2) - 2\sin(x^2) \qquad f''(0) = 0$$
$$f^{(3)}(x) = 8x^3\sin(x^2) - 12x\cos(x^2) \qquad f^{(3)}(0) = 0$$
$$f^{(4)}(x) = 16x^4\cos(x^2) - 12\cos(x^2) + 48x^2\sin(x^2) \qquad f^{(4)}(0) = -12$$
$$\ldots$$

then

$$P(x) = 1 + \frac{0}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-12}{4!}x^4 + \frac{0}{5!}x^5 + \frac{0}{6!}x^6 + \frac{0}{7!}x^7 + \frac{1680}{8!}x^8$$
$$= 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8$$

This is a lot of work.

Task: Find the degree 8 Taylor polynomial for $f(x) = \cos(x^2)$.

- Option 1: Compute lots of derivatives.
- Option 2: Use the Taylor series for cosine.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots$$

$$\cos(x^2) = 1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720} + \cdots$$

$$P(x) = 1 - \frac{x^4}{2} + \frac{x^8}{24} \text{ if we stop at } x^8.$$

much faster

Task: Calculate $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$.

- We already know we can do this with L'Hôpital (3 times!).
- Option 2: Replace $\sin(x)$ with its Taylor series.

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - x}{x^3}$$

$$= \lim_{x\to 0} \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x^3}$$

$$= \lim_{x\to 0} \left(-\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots\right) = \frac{-1}{3!} = \boxed{\frac{-1}{6}}$$